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Why viscous fluids adhere to rugose walls: A mathematical explanation

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Abstract

The main purpose of this paper is to justify rigorously the following assertion: A viscous fluid cannot slip on a wall covered by microscopic asperities because, due to the viscous dissipation, the surface irregularities bring to rest the fluid particles in contact with the wall. In mathematical terms, this corresponds to an asymptotic property established in this paper for any family of fields that slip on oscillating boundaries and remain uniformly bounded in the H^1-norm.

1. Introduction

This paper is devoted to justify rigorously the fact that, asymptotically, a fluid cannot slip on a wall covered by microscopic asperities: the *slip condition*, i.e. the requirement

$$u \cdot n = 0 \quad \text{on the wall,}$$

where u is the velocity and $n = n(x)$ is a normal vector at a boundary point x, which expresses the fact that the wall is not permeable to the fluid particles, provides sufficient information to ensure that, as the size of asperities goes to 0, the fluid

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satisfies the *no-slip condition*, i.e.

$$u = 0 \quad \text{on the wall.}$$

This was noticed and justified for a 2D periodic Stokes flow in [11] and was mathematically proved for a 3D periodic Navier–Stokes flow in [1]. However, the periodicity of the flow at the microscopic scale assumed in these papers is very restrictive. Indeed, it prevents any vortex or any other structure larger than asperities to occur and it implies that the mean velocity over a period is a Couette flow (this enables a satisfactory analysis in this case with a particular proof based on scaling arguments, see [1]).

In the present paper, we will give a mathematical proof of the previous assertion for any 3D flow whatever the governing equation (in fact, no equation is prescribed). This can be viewed as a property of the limit u_0 of a family of vector fields u_ε that slip on a boundary covered by asperities of size ε, with an enstrophy $\int |\nabla u_\varepsilon|^2 \, dx$ that remains bounded as $\varepsilon \to 0$ (Theorem 1).

Roughly speaking, this is due to the fact that sliping with a non-zero velocity dissipates energy on asperities because the direction of velocity suddenly varies as the slope does. For instance, in a 2D domain with a serrated boundary whose slope is alternately $+1$ and -1, if the horizontal velocity is v, then the vertical velocity is alternately $+v$ and $-v$. When the size ε of asperities goes to 0, the energy dissipated by each asperity goes to 0 but not fast enough to compensate the fact that there are many of them. Therefore, the total dissipation grows to infinity and the unique possibility for enstrophy to be uniformly bounded is that the limit velocity vanishes on the wall. A rigorous formulation of this assertion will be given in (8).

We will also prove that our general result applies to a flow governed by the Navier–Stokes equations together with *Navier's law*

$$u \cdot n = 0, \quad (\sigma \cdot n)_{\tan} + \kappa u = 0,$$

where σ denotes the stress tensor and the subscript $_{\tan}$ denotes the tangential component, i.e. $f_{\tan} = f - (f \cdot n)n$ for any vector field f. Of course, the second previous equality means that the friction forces on the wall are proportional to the tangential velocity. Indeed, in this situation the enstrophy remains bounded as $\varepsilon \to 0$ and, therefore, the limit velocity u_0 vanishes on the limit boundary whatever the friction coefficient κ (see Theorem 2). This generalizes, to non-periodic flows, the above-mentioned results of [1,11].

It is worth mentioning that this result is in contradiction with a statement in [8], but the argument used in that reference is false, as we will explain in Remark 5, at the end of Section 4.

Our argument relies on the internal viscous dissipation in the fluid and the geometry of the domain only. It does not require any dissipation of energy due to the friction (or molecular interaction) of the fluid particles in contact with the solid walls.

The effective relative importance of surface roughness and fluid/solid molecular interactions is discussed in [14]. There, the authors show that roughness dominates except for very smooth walls. The reader is referred to [5,7] for an analysis of molecular interaction by molecular dynamics simulation and to [3] for a similar analysis in the case of a two-component fluid.

The flow at the surface of a porous medium is extensively discussed in [6] and references therein. In this case our argument does not apply, since the slip condition $u \cdot n = 0$ is not imposed. In particular, we do not find in the limit the no-slip condition when a rugose interface is modeled by *Fourier's law*

$$\sigma \cdot n + \kappa u = 0 \quad \text{on the wall}$$

(see [2], where a homogeneized friction coefficient κ' is obtained in the limit).

Let us finally mention that many physical and numerical experiments have shown that, when a fluid flows between two plates, the occurrence of asperities on the walls is not irrelevant. In particular, it is known that small *riblets* (tiny asperities parallel to the flow) can be used to reduce considerably the drag experienced by the fluid; see [4,12] and references therein.

This paper is organized as follows. The main result (Theorem 1) is stated and commented in Section 2. It is proved in Section 3. Finally, Section 4 is concerned with the application of Theorem 1 to a viscous fluid near a wall with asperities.

2. Main result

Let us now present our main result with precision. Let $S \subset \mathbb{R}^2$ be a bounded open set and assume that, for each ε with $0 < \varepsilon \leqslant \varepsilon_0$, the function r_ε is given by

$$r_\varepsilon(x') = r_0(x') + \varepsilon\eta\left(\frac{x'}{\varepsilon}\right),$$

where $r_0 \in \mathscr{C}^1(\bar{S})$, $r_0(x') \geqslant a > 0$ and $\eta \in \mathscr{C}^1(\mathbb{R}^2)$ is a periodic function of period (ℓ_1, ℓ_2) in the variable $y' = x'/\varepsilon$. Let G_ε be the open set

$$G_\varepsilon = \{x \in \mathbb{R}^3\colon\ x' \in S,\ 0 < x_3 < r_\varepsilon(x')\}$$

and let us put

$$R_\varepsilon = \{x \in \mathbb{R}^3\colon\ x' \in S,\ x_3 = r_\varepsilon(x')\}$$

(the oscillating piece of boundary). We also set

$$G_0 = \{x \in \mathbb{R}^3\colon\ x' \in S,\ 0 < x_3 < r_0(x')\}$$

(the limit domain) and

$$R_0 = \{x \in \mathbb{R}^3\colon\ x' \in S,\ x_3 = r_0(x')\}.$$

Assume that for each ε we have $u_\varepsilon \in (H^1(G_\varepsilon))^3$, with

$$\int_{G_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \leqslant b, \tag{1}$$

where b is independent of ε. Also, assume that u_0 is a distribution on G_0 such that, as $\varepsilon \to 0$, one has for all $c > 0$

$$u_\varepsilon \to u_0 \quad \text{in } (L^2(\omega_c))^3, \tag{2}$$

where $\omega_c = \{x \in \mathbb{R}^3 \colon x' \in S, \ 0 < x_3 < r_0(x') - c\}$. Finally, assume that η varies in any direction y', at least at one point z', that is

$$\forall y' \in \mathbb{R}^2, \ y' \neq 0, \ \text{there exists } z' \in \mathbb{R}^2 \text{ and } c \in \mathbb{R} \text{ such that } \eta(z' + cy') \neq \eta(z'). \tag{3}$$

Then the following holds:

Theorem 1. *If, for every* $\varepsilon > 0$, *we have*

$$u_\varepsilon \cdot n_\varepsilon = 0 \quad \text{on } R_\varepsilon, \tag{4}$$

then

$$u_0 = 0 \quad \text{on } R_0.$$

Remark 1. The trace of u_0 on R_0 is well defined. Indeed, in view of (1) and (2), we have for all $c > 0$

$$\int_{\omega_c} |\nabla u_0|^2 \, dx \leqslant b,$$

whence $\nabla u_0 \in (L^2(G_0))^{3\times 3}$.

Remark 2. A similar result can be proved in any dimension $N \geqslant 2$. It is also clear that, for this theorem to hold, we only need the hypotheses to be satisfied by a sequence $(u_{\varepsilon_n})_n$, with $\varepsilon_n \to 0$. On the other hand, the result still holds if we replace (1) by the weaker assumption

$$\int_{G_\varepsilon} |\nabla u_\varepsilon|^p \, dx \leqslant b, \tag{5}$$

with $p > 1$. To see this, it suffices to adapt the argument used in Section 3.

Remark 3. If η possesses an invariant direction, i.e., if (3) is not satisfied, the previous result does not hold. More precisely, the arguments used in Section 3 show that, in that case, one of the following two situations is found:

- η is constant; then the unique conclusion is that

$$u_0 \cdot n = 0 \quad \text{on } R_0.$$

Indeed, if such a field u_0 is prescribed, all assumptions are satisfied by the functions $u_\varepsilon(x) = u_0(x_1, x_2, x_3 - \varepsilon\eta)$.

- η possesses only one invariant direction ξ_{inv}; then one has

$$u_0 \cdot n = 0 \quad \text{and} \quad u_0 \cdot \xi_{\text{inv}}^{\perp} = 0 \text{ on } R_0.$$

This is the case of a wall covered with riblets: the fluid possibly slides in the direction ξ_{inv} of the riblets but not in the orthogonal direction.

The invariance of η in the direction ξ_{inv} is equivalent to the fact that η only depends on a scalar variable which is $y' \cdot \xi_{\text{inv}}^{\perp}$; that is equivalent to the existence of a function $\tilde{\eta}$ such that $\eta(y') = \tilde{\eta}(y' \cdot \xi_{\text{inv}}^{\perp})$ for all y'.

Remark 4. The assertions of Theorem 1 and Remark 3 can be gathered together in a single statement in which (3) is not required: whenever the functions u_ε satisfy (1), (2) and (4), one has the following for almost all x in R_0:

$$u_0(x) \in (N(x))^{\perp},$$

where

$$\begin{aligned} N(x) &= \operatorname{Span}\left\{ n(x) - \left(\frac{\partial\eta}{\partial x_1}(y'), \frac{\partial\eta}{\partial x_2}(y'), 0 \right) :\ y' \in (0, l_1) \times (0, l_2) \right\} \\ &= \operatorname{Span}\{n(x), M\} \end{aligned}$$

and

$$M = \operatorname{Span}\left\{ \left(\frac{\partial\eta}{\partial x_1}(y'), \frac{\partial\eta}{\partial x_2}(y'), 0 \right) :\ y' \in (0, l_1) \times (0, l_2) \right\}.$$

In this statement, again (1) can be replaced by (5). Assumption (3) of Theorem 1 (i.e. the fact that η possesses no invariant direction) is equivalent to $\dim M = 2$ and, therefore, to $\dim N(x) = 3$ (since then M is the horizontal plane and $n(x)$ is not horizontal).

The existence of exactly one invariant direction examined in Remark 3 (i.e., the fact that η depends only on one scalar variable) is equivalent to $\dim M = 1$ and therefore to $\dim N(x) = 2$.

The existence of many invariant directions (i.e., the fact that η is constant) is equivalent to $\dim M = 0$ and therefore to $\dim N(x) = 1$.

3. Proof of Theorem 1

In the sequel, C is a generic positive real number that can depend on S, a, b, η and r_0, but not on ε.

First reduction of the problem: The situation is reduced to the case $r_0 \equiv 1$ by means of the change of variable $x \mapsto \hat{x} = (x', 1 + (x_3 - r_0(x'))/a)$ and restriction to the subdomain where $\hat{x}_3 > 0$. Consequently, we will assume from now on that $r_0 \equiv 1$, then, $R_0 = \{(x', 1) \colon x' \in S\}$.

Second reduction of the problem: For each $y' \in \mathbb{R}^2$, we set

$$\lambda(y') = \left(-\frac{\partial \eta}{\partial x_1}(y'), -\frac{\partial \eta}{\partial x_2}(y'), 1 \right).$$

Due to periodicity, η reaches a maximum over $\mathbb{R}^2$, say, at ξ^1. Then $\lambda(\xi^1) = (0, 0, 1)$.

In view of (3), there exist two points ξ^2 and ξ^3 such that $\lambda(\xi^1)$, $\lambda(\xi^2)$ and $\lambda(\xi^3)$ are linearly independent. Indeed, if this were not the case, we would have $\lambda(\xi) = (C\alpha, C\beta, 1)$ for all ξ, for some fixed α and β; thus, we would also have the following, for all y_1 and y_2,

$$\begin{aligned} \frac{d}{dt}\eta(y_1 + t\beta, y_2 - t\alpha) &= \beta \frac{\partial \eta}{\partial x_1}(y_1 + t\beta, y_2 - t\alpha) - \alpha \frac{\partial \eta}{\partial x_2}(y_1 + t\beta, y_2 - t\alpha) \\ &= -C\beta\alpha + C\alpha\beta \\ &= 0, \end{aligned}$$

which is in contradiction with (3). Accordingly, it will be sufficient to prove that, for all $y' \in \mathbb{R}^2$ and almost all $x' \in S$, one has $u_0(x', 1) \cdot \lambda(y') = 0$ or, equivalently,

$$u_0(x', 1) \cdot v(y') = 0, \tag{6}$$

where $v(y') = \lambda(y')/|\lambda(y')|$. Let us denote by Σ the "2D period" of η, *i.e.* the set

$$\Sigma = (0, \ell_1) \times (0, \ell_2),$$

and let K be an arbitrary nonempty compact subset of S. Since $u_0 \in (H^1(G_0))^3$, see Remark 1, a continuous function f_0 is defined on $[0, 1]$ by

$$f_0(x_3) = \int_K \int_\Sigma |u_0(x', x_3) \cdot v(y')|^2 \, dy' \, dx'. \tag{7}$$

To get (6), it will suffice to prove $f(1) = 0$. Since f_0 is continuous, it will be sufficient to prove that

$$\frac{1}{s} \int_{1-2s}^{1-s} f_0(x_3) \, dx_3 \to 0 \quad \text{as } s \to 0. \tag{8}$$

Proof of (8). Let s be given such that $0<s<1/2$. Let us choose $\varepsilon>0$ such that $K+\varepsilon y'\subset S$ for all $y'\in\Sigma$, and such that $\varepsilon\|\eta\|_{L^\infty(\mathbb{R}^2)}<s$. On the other hand, let $(u^n_\varepsilon)_n$ be a sequence in $(\mathscr{C}^1(G_\varepsilon))^3$ converging strongly in $(H^1(G_\varepsilon))^3$ to u_ε. Given $x_3\in(1-2s,1-s)$, $x'\in K$ and $y'\in\Sigma$, we introduce a point $z\in R_\varepsilon$ which is "close" to x by putting

$$z'=x'+\varepsilon y',\quad z_3=1+\varepsilon\eta\left(\frac{z'}{\varepsilon}\right).$$

Then we have

$$\begin{aligned}u^n_\varepsilon(x',x_3)&=u^n_\varepsilon(z',x_3)-\varepsilon\int_0^1 y'\cdot\nabla_{x'}u^n_\varepsilon(x'+t\varepsilon y',x_3)\,dt\\&=u^n_\varepsilon(z',z_3)-\int_{x_3}^{z_3}\frac{\partial u^n_\varepsilon}{\partial x_3}(z',y_3)\,dy_3-\varepsilon\int_0^1 y'\cdot\nabla_{x'}u^n_\varepsilon(x'+t\varepsilon y',x_3)\,dt.\end{aligned}$$

Taking scalar products with $n_\varepsilon(z)$ and using the inequalities $|n_\varepsilon(z)|\leqslant 1$, $|z_3-x_3|\leqslant\varepsilon\eta(z'/\varepsilon)-2s\leqslant C(\varepsilon+s)$ and $|y'|\leqslant C$, we find the following:

$$\begin{aligned}|u^n_\varepsilon(x',x_3)\cdot n_\varepsilon(z)|^2\leqslant C\Bigg(&|u^n_\varepsilon(z',z_3)\cdot n_\varepsilon(z)|^2+(\varepsilon+s)\int_0^{z_3}\left|\frac{\partial u^n_\varepsilon}{\partial x_3}(z',y_3)\right|^2 dy_3\\&+\varepsilon^2\int_0^1|\nabla_{x'}u^n_\varepsilon(x'+t\varepsilon y',x_3)|^2\,dt\Bigg).\end{aligned}$$

Integrating this inequality with respect to x' in K, with respect to y' in Σ and finally with respect to x_3 in $(1-2s,1-s)$, we deduce that

$$\begin{aligned}&\int_{1-2s}^{1-s}\int_\Sigma\int_K|u^n_\varepsilon(x',x_3)\cdot n_\varepsilon(z)|^2\,dx'\,dy'\,dx_3\\&\quad\leqslant Cs\int_\Sigma\int_K|u^n_\varepsilon(z',z_3)\cdot n_\varepsilon(z)|^2\,dx'\,dy'\\&\qquad+\ Cs(\varepsilon+s)\int_\Sigma\int_K\int_0^{z_3}\left|\frac{\partial u^n_\varepsilon}{\partial x_3}(z',y_3)\right|^2 dy_3\,dx'\,dy'\\&\qquad+\ C\varepsilon^2\int_{1-2s}^{1-s}\int_\Sigma\int_K\int_0^1|\nabla_{x'}u^n_\varepsilon(x'+t\varepsilon y',x_3)|^2\,dt\,dx'\,dy'\,dx_3\\&\quad\leqslant Cs\int_\Sigma\int_K|u^n_\varepsilon(z',z_3)\cdot n_\varepsilon(z)|^2\,dx'\,dy'+C(\varepsilon^2+s^2)\int_{G_\varepsilon}|\nabla u^n_\varepsilon(x)|^2\,dx.\end{aligned}$$

The last inequality is implied by the fact that $K+\varepsilon\Sigma\subset S$. Now, taking limits in this inequality as $n\to\infty$, in view of statements (1) and (4) and *Fubini's Theorem*, we find

$$\int_{1-2s}^{1-s}\int_K\int_\Sigma|u_\varepsilon(x',x_3)\cdot n_\varepsilon(z)|^2\,dy'\,dx'\,dx_3\leqslant C(\varepsilon^2+s^2).\tag{9}$$

The normal to R_ε at z is $n_\varepsilon(z) = \nu(z'/\varepsilon)$, *i.e.* $\nu(y' + x'/\varepsilon)$. Since ν is a periodic function and since its "2D period" is Σ, this implies, for almost all (x', σ) in $K \times (s, 2s)$, the identity

$$\int_\Sigma |u_\varepsilon(x', x_3) \cdot n_\varepsilon(z)|^2 \, dy' = \int_\Sigma |u_\varepsilon(x', x_3) \cdot \nu(y')|^2 \, dy'.$$

Then (9) can also be written in the form

$$\int_{1-2s}^{1-s} \int_K \int_\Sigma |u_\varepsilon(x', x_3) \cdot \nu(y')|^2 \, dy' \, dx' \, dx_3 \leqslant C(\varepsilon^2 + s^2).$$

Taking limits as $\varepsilon \to 0$, we obtain

$$\frac{1}{s} \int_{1-2s}^{1-s} \int_K \int_\Sigma |u_0(x', x_3) \cdot \nu(y')|^2 \, dy' \, dx' \, dx_3 \leqslant Cs.$$

Consequently, we have proved (8). This ends the proof of Theorem 1. $\square$

4. A consequence: the asymptotic behavior of a viscous fluid near a wall with asperities

Theorem 1 can be used to identify the limit of the solution of the stationary Navier–Stokes system satisfying Navier's law on an oscillating boundary. In order to fix ideas, let us introduce the fluid domains Ω_ε and Ω_0, with

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3\colon\ 0 < x_3 < r_\varepsilon(x')\}$$

and

$$\Omega_0 = \{x \in \mathbb{R}^3\colon\ 0 < x_3 < \ell_3\}.$$

Here, r_ε is given by

$$r_\varepsilon(x') = \ell_3 + \varepsilon\eta\left(\frac{x'}{\varepsilon}\right)$$

(ℓ_3 is positive and constant) and $\eta \in \mathscr{C}^1(\mathbb{R}^2)$ is periodic of period (ℓ_1, ℓ_2) in the variable $y' = x'/\varepsilon$. We set

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^3\colon\ x_3 = r_\varepsilon(x')\}$$

(the upper boundary of Ω_ε), and

$$\Gamma_0 = \{x \in \mathbb{R}^3\colon\ x_3 = \ell_3\}, \quad P = \{x \in \mathbb{R}^3\colon\ x_3 = 0\}.$$

Let us consider the stationary Navier–Stokes system in Ω_ε

$$-\nu\Delta u_\varepsilon + (u_\varepsilon \cdot \nabla)u_\varepsilon + \nabla p_\varepsilon = 0, \quad \nabla \cdot u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \tag{10}$$

completed with the *slip* and *friction* conditions

$$u_\varepsilon \cdot n_\varepsilon = 0, \quad (\sigma_\varepsilon \cdot n_\varepsilon)_{\tan} + \kappa u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon \tag{11}$$

(n_ε is the unit normal vector on Γ_ε and σ_ε is the stress tensor associated to $(u_\varepsilon, p_\varepsilon)$),

$$u_\varepsilon \cdot n = 0, \quad (\sigma_\varepsilon \cdot n)_{\tan} + \kappa(u_\varepsilon - g) = 0 \quad \text{on } P \tag{12}$$

(g is a non-zero vector of the form $g = (g_1, g_2, 0)$) and the following additional condition:

$$(u_\varepsilon, p_\varepsilon) \text{ is } x'\text{-periodic, of period } (\ell_1, \ell_2). \tag{13}$$

Let L be given by

$$L = \max(\ell_1, \ell_2, \ell_3)$$

(a characteristic length of Ω_0) and let us introduce the associated Reynolds number

$$\mathrm{Re} = \frac{L|g|}{\nu}.$$

For simplicity, we assume that Re is sufficiently small. Then, system (10)–(13) possesses exactly one solution

$$(u_\varepsilon, p_\varepsilon) \in (H^1_{\mathrm{loc}}(\Omega_\varepsilon))^3 \times L^2_{\mathrm{loc}}(\Omega_\varepsilon).$$

satisfying

$$\int_{\Omega_\varepsilon \cap \{|x'|<K\}} |\nabla u_\varepsilon|^2\, dx + \int_{\Omega_\varepsilon \cap \{|x'|<K\}} |u_\varepsilon|^2\, dx \leqslant b_K \tag{14}$$

for all $K > 0$, where b_K is independent of ε (the proof of this assertion is essentially given in Refs. [1,2]). From (14), it is not difficult to deduce the existence of a function $u_0 \in (H^1_{\mathrm{loc}}(\Omega_0))^3$ such that, at least for a subsequence, we have

$$u_\varepsilon \to u_0 \text{ weakly in } (H^1_{\mathrm{loc}}(\omega_c))^3 \text{ and strongly in } (L^2_{\mathrm{loc}}(\omega_c))^3$$

for all $c > 0$, where $\omega_c = \{x \in \mathbb{R}^3 \colon 0 < x_3 < \ell_3 - c\}$.

Then, as a consequence of Theorem 1, we obtain the following:

Theorem 2. *Assume that* Re *is sufficiently small,* η *satisfies* (3) *and* $\varepsilon \to 0$. *Then* u_ε *converges to* u_0, *i.e., together with some* p_0, *the unique solution to the stationary Navier–Stokes equations*

$$-\nu\Delta u_0 + (u_0 \cdot \nabla)u_0 + \nabla p_0 = 0, \quad \nabla \cdot u_0 = 0 \quad \text{in } \Omega_0,$$

completed with the boundary conditions

$$u_0 = 0 \quad on \ \Gamma_0$$

and

$$u_0 \cdot n = 0, \quad (\sigma_0 \cdot n)_{\tan} + \kappa(u_0 - g) = 0 \quad on \ P$$

and the periodicity requirement

$$(u_0, p_0) \text{ is } x'\text{-periodic, of period } (\ell_1, \ell_2). \qquad \square$$

Notice that, in this simple case, u_0 and p_0 can be computed explicitly. Indeed, one has

$$u_0 = \frac{\kappa(\ell_3 - x_3)^2}{(2v + \kappa)\ell_3^2} g, \quad p_0 = \frac{2v\kappa}{(2v + \kappa)\ell_3^2}(g_1 x_1 + g_2 x_2) \quad \text{in } \Omega_0 \tag{15}$$

(as usual, p_0 is defined up to an additive constant). The convergence of u_ε towards u_0 provides a rigorous justification of the fact that a viscous fluid cannot slip on a wall with too many asperities.

Remark 5. As we have already indicated, our results are in contradiction with a result in [8]. In that paper, the oscillations are described in a slightly different way, but everything can be adapted to our context. A consequence of Theorem 2 in [8] is that, in the previous situation, at least when Re is sufficiently small, the limit velocity field should satisfy a friction condition on Γ_0 of the form

$$(\sigma_0 \cdot n)_{\tan} + \kappa' u_0 = 0 \quad \text{on } \Gamma_0$$

for some $\kappa' > \kappa$. But this is false in view of (15).

The wrong point in the proof of Theorem 2 in [8] is the following. Near the end of the proof, given a function v_0 satisfying

$$v_0 \in (H^1(\Omega_0 \cap X))^3, \quad \nabla \cdot v_0 = 0 \quad \text{in } \Omega_0 \cap X, \quad v_0 \cdot n = 0 \quad \text{on } \Gamma_0 \cap X,$$

where $X = \{x \in \mathbb{R}^3 \colon 0 < x_1 < \ell_1, \ 0 < x_2 < \ell_2\}$, the author claims (but does not prove) that it may be approached by functions v_ε such that

$$v_\varepsilon \in (H^1(\Omega_\varepsilon \cap X))^3, \quad \nabla \cdot v_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon \cap X, \quad v_\varepsilon \cdot n_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon \cap X,$$

which converge weakly to v_0 in the H^1_{loc} sense and satisfy

$$\int_{\Gamma_\varepsilon \cap X} |v_\varepsilon|^2 \, d\Gamma \to \int_{\Gamma_0 \cap X} |v_0|^2 \, d\Gamma \quad \text{as } \varepsilon \to 0$$

and

$$\int_{\Omega_\varepsilon \cap X} |\nabla v_\varepsilon|^2 \, dx \to \int_{\Omega_0 \cap X} |\nabla v_0|^2 \, dx \quad \text{as } \varepsilon \to 0.$$

But, in view of Theorem 1, the limit v_0 of such a v_ε must vanish on $\Gamma_0 \cap X$. Therefore, if v_0 does not vanish on $\Gamma_0 \cap X$, then such functions v_ε cannot exist and the proof of [8] fails.

Remark 6. When a viscous fluid, like air or water, moves at high speed past a wall at rest, the fluid adheres to the wall and, close to the wall, a thin boundary layer appears in which the velocity field changes sudddenly in the normal direction, see for instance [10] or [13]. In these cases, in order to avoid the (complicate) description and/or computation of the flow variables in such a boundary layer, the no-slip condition on the wall is frequently replaced by the Navier law (11) with a friction coefficient κ depending on the rugosity of the wall (and possibly on u). A review of mathematical results in that direction can be found in [9].

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References

[1] Y. Amirat, D. Bresch, J. Lemoine, J. Simon, Effect of rugosity on a flow governed by stationary Navier–Stokes equations, Quart. Appl. Math. 59 (4) (2001) 768–785.

[2] Y. Amirat, B. Climent, E. Fernández-Cara, J. Simon, The Stokes equations with Fourier boundary conditions on a wall with asperities, Math. Models Methods Appl. Sci. 24 (2001) 255–276.

[3] H. Brenner, V. Ganesan, Molecular effects: are conditions at the boundary "boundary conditions"?, Phys. Rev. 61 (6) (2000) 6879–7547.

[4] D.C. Chu, G.E. Karniadakis, A direct numerical simulation of laminar and turbulent flow over riblet-mounted surfaces, J. Fluid Mech. 250 (1993) 1–42.

[5] M. Cieplak, J. Koplik, J.R. Banavar, Boundary conditions at a fluid–solid interface, Phys. Rev. Lett. 86 (2001) 803–806.

[6] D.F. James, A.M.J. Davis, Flow at the interface of a model fibrous porous medium, J. Fluid Mech. 426 (2001) 47–72.

[7] J. Koplik, J.R. Banavar, Continuum deductions from molecular hydrodynamics, Annu. Rev. Fluid Mech. 27 (1995) 257–292.

[8] E. Marusić-Paloka, Average of the Navier's law on the rapidly oscillating boundary, J. Math. Anal. Appl. 259 (2001) 685–701.
[9] A. Mikelić, Recent developments in multiscale problems coming from fluid mechanics, Publications du Laboratoire d'Analyse Numérique, Université Lyon 1, 2001, in Trends in Nonlinear Analysis, Springer, Berlin, to appear.
[10] R.L. Panton, Incompressible Flow, Wiley-Interscience, New York, 1984.
[11] S. Richardson, On the no-slip boundary condition, J. Fluid Mech. 59 (1973) 707–719.
[12] A.M. Savill, T.V. Truong, I.L. Ryhming, Turbulent drag reduction by passive means: a review and report on the First European Drag Reduction Meeting, J. Méc. Théor. Appl. 7 (1988) 353–378.
[13] H. Schlichting, Boundary-Layer Theory, McGraw-Hill, New York, 1968.
[14] Y. Zhu, S. Granick, Limits of the hydrodynamic no-slip boundary condition, Phys. Rev. Lett. 88 (2002) 1061021–1061024.